Basic Properties of Real Numbers: Exponents*

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Abstract

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The symbols, notations, and properties of numbers that form the basis of algebra, as well as exponents and the rules of exponents, are introduced in this chapter. Each property of real numbers and the rules of exponents are expressed both symbolically and literally. Literal explanations are included because symbolic explanations alone may be difficult for a student to interpret. Objectives of this module: understand exponential notation, be able to read exponential notation, understand how to use exponential notation with the order of operations.

1 Overview

- Exponential Notation
- Reading Exponential Notation
- The Order of Operations

2 Exponential Notation

In Section here[1] we were reminded that multiplication is a description for repeated addition. A natural question is "Is there a description for **repeated** multiplication?" The answer is yes. The notation that describes repeated multiplication is **exponential notation**.

Factors

In multiplication, the numbers being multiplied together are called **factors**. In repeated multiplication, all the factors are the same. In nonrepeated multiplication, none of the factors are the same. For example,

Example 1

$18 \cdot 18 \cdot 18 \cdot 18$	Repeated multiplication of 18. All four factors, 18, are the same.
$x \cdot x \cdot x \cdot x \cdot x$	Repeated multiplication of x. All five factors, x, are the same.
$3 \cdot 7 \cdot a$	Nonrepeated multiplication. None of the factors are the same.

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[1] "Basic Properties of Real Numbers: Properties of the Real Numbers" <http://cnx.org/content/m21894/latest/>

Exponential notation is used to show repeated multiplication of the same factor. The notation consists of using a **superscript on the factor that is repeated**. The superscript is called an **exponent**.

Exponential Notation

If x is any real number and n is a natural number, then

$$x^n = \underbrace{x \cdot x \cdot x \cdot ... \cdot x}_{n \text{ factors of } x}$$

An exponent records the number of identical factors in a multiplication.

Note that the definition for exponential notation only has meaning for natural number exponents. We will extend this notation to include other numbers as exponents later.

3 Sample Set A

Example 2

$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^6$.

The repeated factor is 7. The exponent 6 records the fact that 7 appears 6 times in the multiplication.

Example 3

$x \cdot x \cdot x \cdot x = x^4$.

The repeated factor is x. The exponent 4 records the fact that x appears 4 times in the multiplication.

Example 4

$(2y)(2y)(2y) = (2y)^3$.

The repeated factor is $2y$. The exponent 3 records the fact that the factor $2y$ appears 3 times in the multiplication.

Example 5

$2yyy = 2y^3$.

The repeated factor is y. The exponent 3 records the fact that the factor y appears 3 times in the multiplication.

Example 6

$(a+b)(a+b)(a-b)(a-b)(a-b) = (a+b)^2(a-b)^3$.

The repeated factors are $(a+b)$ and $(a-b)$, $(a+b)$ appearing 2 times and $(a-b)$ appearing 3 times.

4 Practice Set A

Write each of the following using exponents.

Exercise 1 *(Solution on p. 10.)*

$a \cdot a \cdot a \cdot a$

Exercise 2 *(Solution on p. 10.)*

$(3b)(3b)(5c)(5c)(5c)(5c)$

Exercise 3 *(Solution on p. 10.)*

$2 \cdot 2 \cdot 7 \cdot 7 \cdot 7 \cdot (a-4)(a-4)$

Exercise 4 *(Solution on p. 10.)*

$8xxxyzzzzz$

CAUTION
It is extremely important to realize and remember that an exponent applies only to the factor to which it is directly connected.

5 Sample Set B

Example 7
$8x^3$ means $8 \cdot xxx$ and **not**$8x8x8x$. The exponent 3 applies only to the factor x since it is only to the factor x that the 3 is connected.

Example 8
$(8x)^3$ means $(8x)(8x)(8x)$ since the parentheses indicate that the exponent 3 is directly connected to the factor $8x$. Remember that the grouping symbols ( ) indicate that the quantities inside are to be considered as one single number.

Example 9
$34(a+1)^2$ means $34 \cdot (a+1)(a+1)$ since the exponent 2 applies only to the factor $(a+1)$.

6 Practice Set B

Write each of the following without exponents.

Exercise 5 *(Solution on p. 10.)*
$4a^3$

Exercise 6 *(Solution on p. 10.)*
$(4a)^3$

7 Sample Set C

Example 10
Select a number to show that $(2x)^2$ is not always equal to $2x^2$.

Suppose we choose x to be 5. Consider both $(2x)^2$ and $2x^2$.

$$\begin{array}{ccc} (2x)^2 & & 2x^2 \\ (2 \cdot 5)^2 & & 2 \cdot 5^2 \\ (10)^2 & & 2 \cdot 25 \\ 100 & \neq & 50 \end{array} \quad (1)$$

Notice that $(2x)^2 = 2x^2$ only when $x = 0$.

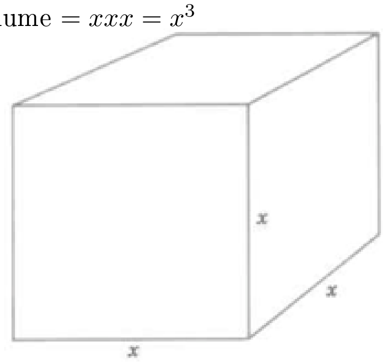
8 Practice Set C

Exercise 7 *(Solution on p. 10.)*
Select a number to show that $(5x)^2$ is not always equal to $5x^2$.

9 Reading Exponential Notation

In x^n,
Base
x is the **base**
Exponent
n is the **exponent**
Power
The number represented by x^n is called a **power**.
x to the nth Power
The term x^n is read as "x to the n th power," or more simply as "x to the nth."
x Squared and x Cubed
The symbol x^2 is often read as "x squared," and x^3 is often read as "x cubed." A natural question is "Why are geometric terms appearing in the exponent expression?" The answer for x^3 is this: x^3 means $x \cdot x \cdot x$. In geometry, the volume of a rectangular box is found by multiplying the length by the width by the depth. A cube has the same length on each side. If we represent this length by the letter x then the volume of the cube is $x \cdot x \cdot x$, which, of course, is described by x^3. (Can you think of why x^2 is read as x squared?)

Cube with
length $= x$
width $= x$
depth $= x$
Volume $= xxx = x^3$



10 The Order of Operations

In Section here[2] we were introduced to the order of operations. It was noted that we would insert another operation before multiplication and division. We can do that now.

The Order of Operations

1. Perform all operations inside grouping symbols beginning with the innermost set.
2. Perform all exponential **operations as you come to** them, moving left-to-right.
3. Perform all multiplications and divisions as you come to them, moving left-to-right.
4. Perform all additions and subtractions as you come to them, moving left-to-right.

11 Sample Set D

Use the order of operations to simplify each of the following.

[2]"Basic Properties of Real Numbers: Symbols and Notations" <http://cnx.org/content/m18872/latest/>

Example 11

$2^2 + 5 = 4 + 5 = 9$

Example 12

$5^2 + 3^2 + 10 = 25 + 9 + 10 = 44$

Example 13

$$\begin{aligned} 2^2 + (5)(8) - 1 &= 4 + (5)(8) - 1 \\ &= 4 + 40 - 1 \\ &= 43 \end{aligned}$$

Example 14

$$\begin{aligned} 7 \cdot 6 - 4^2 + 1^5 &= 7 \cdot 6 - 16 + 1 \\ &= 42 - 16 + 1 \\ &= 27 \end{aligned}$$

Example 15

$$\begin{aligned} (2+3)^3 + 7^2 - 3(4+1)^2 &= (5)^3 + 7^2 - 3(5)^2 \\ &= 125 + 49 - 3(25) \\ &= 125 + 49 - 75 \\ &= 99 \end{aligned}$$

Example 16

$$\begin{aligned} \left[4(6+2)^3\right]^2 &= \left[4(8)^3\right]^2 \\ &= [4(512)]^2 \\ &= [2048]^2 \\ &= 4,194,304 \end{aligned}$$

Example 17

$$\begin{aligned} 6\left(3^2 + 2^2\right) + 4^2 &= 6(9+4) + 4^2 \\ &= 6(13) + 4^2 \\ &= 6(13) + 16 \\ &= 78 + 16 \\ &= 94 \end{aligned}$$

Example 18

$$\begin{aligned} \frac{6^2+2^2}{4^2+6\cdot 2^2} + \frac{1^3+8^2}{10^2-(19)(5)} &= \frac{36+4}{16+6\cdot 4} + \frac{1+64}{100-95} \\ &= \frac{36+4}{16+24} + \frac{1+64}{100-95} \\ &= \frac{40}{40} + \frac{65}{5} \\ &= 1 + 13 \\ &= 14 \end{aligned}$$

12 Practice Set D

Use the order of operations to simplify the following.

Exercise 8 ***(Solution on p. 10.)***

$3^2 + 4 \cdot 5$

Exercise 9 ***(Solution on p. 10.)***

$2^3 + 3^3 - 8 \cdot 4$

Exercise 10 ***(Solution on p. 10.)***

$1^4 + \left(2^2 + 4\right)^2 \div 2^3$

Exercise 11 ***(Solution on p. 10.)***

$\left[6\left(10 - 2^3\right)\right]^2 - 10^2 - 6^2$

Exercise 12 ***(Solution on p. 10.)***

$\frac{5^2+6^2-10}{1+4^2} + \frac{0^4-0^5}{7^2-6\cdot2^3}$

13 Exercises

For the following problems, write each of the quantities using exponential notation.

Exercise 13 ***(Solution on p. 10.)***

b to the fourth

Exercise 14

a squared

Exercise 15 ***(Solution on p. 10.)***

x to the eighth

Exercise 16

(-3) cubed

Exercise 17 ***(Solution on p. 10.)***

5 times s squared

Exercise 18

3 squared times y to the fifth

Exercise 19 ***(Solution on p. 10.)***

a cubed minus $(b + 7)$ squared

Exercise 20

$(21 - x)$ cubed plus $(x + 5)$ to the seventh

Exercise 21 ***(Solution on p. 10.)***

$xxxxx$

Exercise 22

$(8)\,(8)\,xxxx$

Exercise 23 ***(Solution on p. 10.)***

$2 \cdot 3 \cdot 3 \cdot 3 \cdot 3xxyyyyy$

Exercise 24

$2 \cdot 2 \cdot 5 \cdot 6 \cdot 6 \cdot 6xyyzzzwwww$

Exercise 25 ***(Solution on p. 10.)***

$7xx\,(a + 8)\,(a + 8)$

Exercise 26

$10xyy\,(c + 5)\,(c + 5)\,(c + 5)$

Exercise 27 *(Solution on p. 10.)*

$4x4x4x4x4x$

Exercise 28

$(9a)(9a)(9a)(9a)$

Exercise 29 *(Solution on p. 10.)*

$(-7)(-7)(-7)aabbba(-7)baab$

Exercise 30

$(a-10)(a-10)(a+10)$

Exercise 31 *(Solution on p. 10.)*

$(z+w)(z+w)(z+w)(z-w)(z-w)$

Exercise 32

$(2y)(2y)2y2y$

Exercise 33 *(Solution on p. 10.)*

$3xyxxy-(x+1)(x+1)(x+1)$

For the following problems, expand the quantities so that no exponents appear.

Exercise 34

4^3

Exercise 35 *(Solution on p. 10.)*

6^2

Exercise 36

7^3y^2

Exercise 37 *(Solution on p. 10.)*

$8x^3y^2$

Exercise 38

$\left(18x^2y^4\right)^2$

Exercise 39 *(Solution on p. 11.)*

$\left(9a^3b^2\right)^3$

Exercise 40

$5x^2\left(2y^3\right)^3$

Exercise 41 *(Solution on p. 11.)*

$10a^3b^2(3c)^2$

Exercise 42

$(a+10)^2\left(a^2+10\right)^2$

Exercise 43 *(Solution on p. 11.)*

$\left(x^2-y^2\right)\left(x^2+y^2\right)$

For the following problems, select a number (or numbers) to show that

Exercise 44

$(5x)^2$ is not generally equal to $5x^2$.

Exercise 45 *(Solution on p. 11.)*

$(7x)^2$ is not generally equal to $7x^2$.

Exercise 46

$(a+b)^2$ is not generally equal to a^2+b^2.

Exercise 47 *(Solution on p. 11.)*

For what real number is $(6a)^2$ equal to $6a^2$?

Exercise 48

For what real numbers, a and b, is $(a+b)^2$ equal to a^2+b^2?

Use the order of operations to simplify the quantities for the following problems.

Exercise 49 *(Solution on p. 11.)*

3^2+7

Exercise 50

4^3-18

Exercise 51 *(Solution on p. 11.)*

$5^2+2\left(40\right)$

Exercise 52

$8^2+3+5\left(2+7\right)$

Exercise 53 *(Solution on p. 11.)*

$2^5+3\left(8+1\right)$

Exercise 54

$3^4+2^4(1+5)^3$

Exercise 55 *(Solution on p. 11.)*

$\left(6^2-4^2\right)\div 5$

Exercise 56

$2^2\left(10-2^3\right)$

Exercise 57 *(Solution on p. 11.)*

$\left(3^4-4^3\right)\div 17$

Exercise 58

$(4+3)^2+1\div\left(2\cdot 5\right)$

Exercise 59 *(Solution on p. 11.)*

$\left(2^4+2^5-2^3\cdot 5\right)^2\div 4^2$

Exercise 60

$1^6+0^8+5^2(2+8)^3$

Exercise 61 *(Solution on p. 11.)*

$\left(7\right)\left(16\right)-9^2+4\left(1^1+3^2\right)$

Exercise 62

$\frac{2^3-7}{5^2}$

Exercise 63 *(Solution on p. 11.)*

$\frac{(1+6)^2+2}{19}$

Exercise 64

$\frac{6^2-1}{5}+\frac{4^3+(2)(3)}{10}$

Exercise 65 *(Solution on p. 11.)*

$\frac{5\left[8^2-9(6)\right]}{2^5-7}+\frac{7^2-4^2}{2^4-5}$

Exercise 66

$\frac{(2+1)^3+2^3+1^3}{6^2}-\frac{15^2-[2(5)]^2}{5\cdot 5^2}$

Exercise 67 *(Solution on p. 11.)*

$\frac{6^3-2\cdot 10^2}{2^2}+\frac{18\left(2^3+7^2\right)}{2(19)-3^3}$

14 Exercises for Review

Exercise 68
(**here**[3]) Use algebraic notation to write the statement "a number divided by eight, plus five, is equal to ten."

Exercise 69 ***(Solution on p. 11.)***
(**here**[4]) Draw a number line that extends from -5 to 5 and place points at all real numbers that are strictly greater than -3 but less than or equal to 2.

Exercise 70
(**here**[5]) Is every integer a whole number?

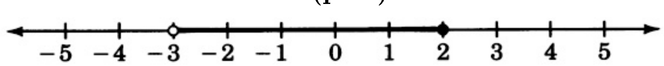
Exercise 71 ***(Solution on p. 11.)***
(**here**[6]) Use the commutative property of multiplication to write a number equal to the number yx.

Exercise 72
(**here**[7]) Use the distributive property to expand $3(x+6)$.

[3]"Basic Properties of Real Numbers: Symbols and Notations" <http://cnx.org/content/m18872/latest/>
[4]"Basic Properties of Real Numbers: The Real Number Line and the Real Numbers" <http://cnx.org/content/m21895/latest/>
[5]"Basic Properties of Real Numbers: The Real Number Line and the Real Numbers" <http://cnx.org/content/m21895/latest/>
[6]"Basic Properties of Real Numbers: Properties of the Real Numbers" <http://cnx.org/content/m21894/latest/>
[7]"Basic Properties of Real Numbers: Properties of the Real Numbers" <http://cnx.org/content/m21894/latest/>

Solutions to Exercises in this Module

Solution to Exercise 1 (p. 2)
a^4
Solution to Exercise 2 (p. 2)
$(3b)^2(5c)^4$
Solution to Exercise 3 (p. 2)
$2^2 \cdot 7^3(a-4)^2$
Solution to Exercise 4 (p. 2)
$8x^3yz^5$
Solution to Exercise 5 (p. 3)
$4aaa$
Solution to Exercise 6 (p. 3)
$(4a)(4a)(4a)$
Solution to Exercise 7 (p. 3)
Select $x = 3$. Then $(5 \cdot 3)^2 = (15)^2 = 225$, but $5 \cdot 3^2 = 5 \cdot 9 = 45$. $225 \neq 45$.
Solution to Exercise 8 (p. 6)
29
Solution to Exercise 9 (p. 6)
3
Solution to Exercise 10 (p. 6)
9
Solution to Exercise 11 (p. 6)
8
Solution to Exercise 12 (p. 6)
3
Solution to Exercise 13 (p. 6)
b^4
Solution to Exercise 15 (p. 6)
x^8
Solution to Exercise 17 (p. 6)
$5s^2$
Solution to Exercise 19 (p. 6)
$a^3 - (b+7)^2$
Solution to Exercise 21 (p. 6)
x^5
Solution to Exercise 23 (p. 6)
$2\left(3^4\right)x^2y^5$
Solution to Exercise 25 (p. 6)
$7x^2(a+8)^2$
Solution to Exercise 27 (p. 6)
$(4x)^5$ or 4^5x^5
Solution to Exercise 29 (p. 7)
$(-7)^4a^5b^5$
Solution to Exercise 31 (p. 7)
$(z+w)^3(z-w)^2$
Solution to Exercise 33 (p. 7)
$3x^3y^2 - (x+1)^3$
Solution to Exercise 35 (p. 7)
$6 \cdot 6$

Solution to Exercise 37 (p. 7)
$8 \cdot x \cdot x \cdot x \cdot y \cdot y$
Solution to Exercise 39 (p. 7)
$(9aaabb)(9aaabb)(9aaabb)$ or $9 \cdot 9 \cdot 9aaaaaaaaabbbbbb$
Solution to Exercise 41 (p. 7)
$10aaabb(3c)(3c)$ or $10 \cdot 3 \cdot 3aaabbcc$
Solution to Exercise 43 (p. 7)
$(xx - yy)(xx + yy)$
Solution to Exercise 45 (p. 7)
Select $x = 2$. Then, $196 \neq 28$.
Solution to Exercise 47 (p. 7)
zero
Solution to Exercise 49 (p. 8)
16
Solution to Exercise 51 (p. 8)
105
Solution to Exercise 53 (p. 8)
59
Solution to Exercise 55 (p. 8)
4
Solution to Exercise 57 (p. 8)
1
Solution to Exercise 59 (p. 8)
4
Solution to Exercise 61 (p. 8)
71
Solution to Exercise 63 (p. 8)
$\frac{51}{19}$
Solution to Exercise 65 (p. 8)
5
Solution to Exercise 67 (p. 8)
$\frac{1070}{11}$ or $97.\overline{27}$
Solution to Exercise 69 (p. 9)


Solution to Exercise 71 (p. 9)
xy