Basic Properties of Real Numbers: The Real Number Line and the Real Numbers*

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Abstract

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The symbols, notations, and properties of numbers that form the basis of algebra, as well as exponents and the rules of exponents, are introduced in this chapter. Each property of real numbers and the rules of exponents are expressed both symbolically and literally. Literal explanations are included because symbolic explanations alone may be difficult for a student to interpret. Objectives of this module: be familiar with the real number line and the real numbers, understand the ordering of the real numbers.

1 Overview

- The Real Number Line
- The Real Numbers
- Ordering the Real Numbers

2 The Real Number Line

Real Number Line
In our study of algebra, we will use several collections of numbers. The **real number line** allows us to **visually** display the numbers in which we are interested.

A line is composed of infinitely many points. To each point we can associate a unique number, and with each number we can associate a particular point.

Coordinate
The number associated with a point on the number line is called the **coordinate** of the point.

Graph
The point on a line that is associated with a particular number is called the **graph** of that number.

We construct the real number line as follows:

Construction of the Real Number Line

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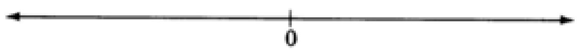
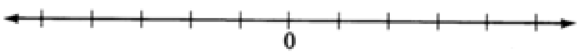
1. Draw a horizontal line.

2. Choose any point on the line and label it 0. This point is called the **origin**.

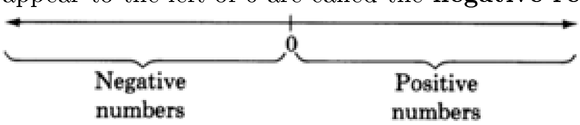


3. Choose a convenient length. This length is called "1 unit." Starting at 0, mark this length off in both directions, being careful to have the lengths look like they are about the same.



We now define a real number.

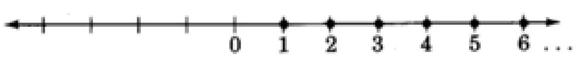
Real Number
A **real number** is any number that is the coordinate of a point on the real number line.
Positive and Negative Real Numbers
The collection of these infinitely many numbers is called the **collection of real numbers**. The real numbers whose graphs are to the right of 0 are called the **positive real numbers**. The real numbers whose graphs appear to the left of 0 are called the **negative real numbers**.



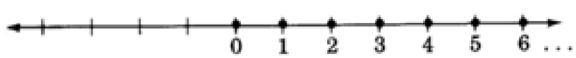
The number 0 is neither positive nor negative.

3 The Real Numbers

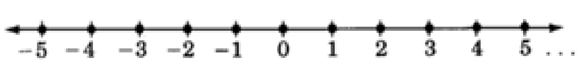
The collection of real numbers has many subcollections. The subcollections that are of most interest to us are listed below along with their notations and graphs.
Natural Numbers
The **natural numbers** (N)**:** $\{1, 2, 3, \ldots\}$



Whole Numbers
The **whole numbers** (W)**:** $\{0, 1, 2, 3, \ldots\}$



Notice that every natural number is a whole number.
Integers
The **integers** (Z)**:** $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$



Notice that every whole number is an integer.
Rational Numbers
The **rational numbers** (Q)**:** Rational numbers are real numbers that can be written in the form a/b, where a and b are integers, and $b \neq 0$.

Fractions
Rational numbers are commonly called **fractions.**
Division by 1
Since b can equal 1, every integer is a rational number: $\frac{a}{1} = a$.
Division by 0
Recall that $10/2 = 5$ since $2 \cdot 5 = 10$. However, if $10/0 = x$, then $0 \cdot x = 10$. But $0 \cdot x = 0$, not 10. This suggests that no quotient exists.

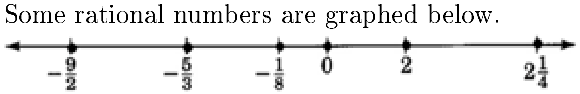
Now consider $0/0 = x$. If $0/0 = x$, then $0 \cdot x = 0$. But this means that x could be any number, that is, $0/0 = 4$ since $0 \cdot 4 = 0$, or $0/0 = 28$ since $0 \cdot 28 = 0$. This suggests that the quotient is indeterminant.
$x/0$ **Is Undefined or Indeterminant**
Division by 0 is undefined or indeterminant.

Do not divide by 0.

Rational numbers have decimal representations that either terminate or do not terminate but contain a repeating block of digits. Some examples are:

$$\underbrace{\frac{3}{4} = 0.75}_{\text{Terminating}} \qquad \underbrace{\frac{15}{11} = 1.36363636\ldots}_{\text{Nonterminating, but repeating}}$$

Some rational numbers are graphed below.



Irrational Numbers
The **irrational numbers** (Ir)**:** Irrational numbers are numbers that cannot be written as the quotient of two integers. They are numbers whose decimal representations are nonterminating and nonrepeating. Some examples are

$4.01001000100001\ldots \quad \pi = 3.1415927\ldots$

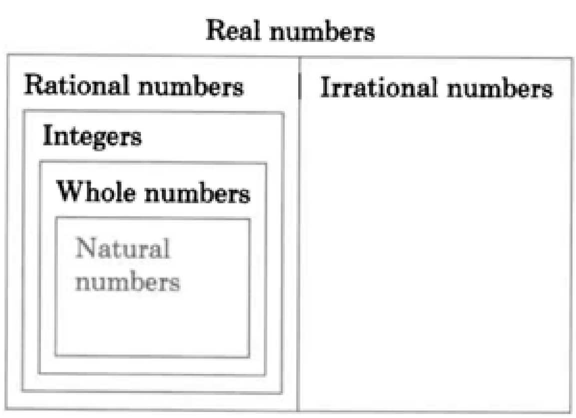
Notice that the collections of rational numbers and irrational numbers have no numbers in common.

When graphed on the number line, the rational and irrational numbers account for every point on the number line. Thus each point on the number line has a coordinate that is either a rational or an irrational number.

In summary, we have

4 Sample Set A

The summaray chart illustrates that



Example 1
Every natural number is a real number.

Example 2
Every whole number is a real number.

Example 3
No integer is an irrational number.

5 Practice Set A

Exercise 1 *(Solution on p. 9.)*
Is every natural number a whole number?

Exercise 2 *(Solution on p. 9.)*
Is every whole number an integer?

Exercise 3 *(Solution on p. 9.)*
Is every integer a rational number?

Exercise 4 *(Solution on p. 9.)*
Is every rational number a real number?

Exercise 5 *(Solution on p. 9.)*
Is every integer a natural number?

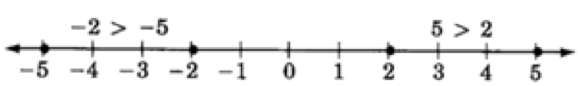
Exercise 6 *(Solution on p. 9.)*
Is there an integer that is a natural number?

6 Ordering the Real Numbers

Ordering the Real Numbers
A real number b is said to be greater than a real number a, denoted $b > a$, if the graph of b is to the right of the graph of a on the number line.

7 Sample Set B

As we would expect, $5 > 2$ since 5 is to the right of 2 on the number line. Also, $-2 > -5$ since -2 is to the right of -5 on the number line.



8 Practice Set B

Exercise 7 *(Solution on p. 9.)*
Are all positive numbers greater than 0?

Exercise 8 *(Solution on p. 9.)*
Are all positive numbers greater than all negative numbers?

Exercise 9 *(Solution on p. 9.)*
Is 0 greater than all negative numbers?

Exercise 10 *(Solution on p. 9.)*
Is there a largest positive number? Is there a smallest negative number?

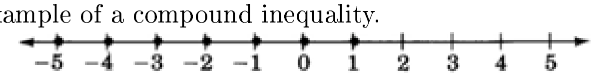
Exercise 11 *(Solution on p. 9.)*
How many real numbers are there? How many real numbers are there between 0 and 1?

9 Sample Set C

Example 4
What integers can replace x so that the following statement is true?

$$-4 \le x < 2$$

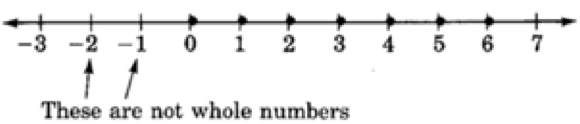
This statement indicates that the number represented by x is between -4 and 2. Specifically, -4 is less than or equal to x, and at the same time, x is strictly less than 2. This statement is an example of a compound inequality.



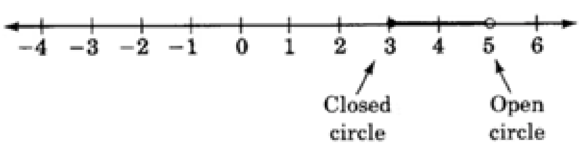
The integers are $-4, \ -3, \ -2, \ -1, 0, 1$.

Example 5
Draw a number line that extends from -3 to 7. Place points at all whole numbers between and including -2 and 6.

Example 6
Draw a number line that extends from -4 to 6 and place points at all real numbers greater than or equal to 3 but strictly less than 5.



It is customary to use a **closed circle** to indicate that a point is included in the graph and an **open circle** to indicate that a point is not included.



10 Practice Set C

Exercise 12 *(Solution on p. 9.)*
What whole numbers can replace x so that the following statement is true?

$-3 \le x < 3$

Exercise 13 *(Solution on p. 9.)*
Draw a number line that extends from -5 to 3 and place points at all numbers greater than or equal to -4 but strictly less than 2.

11 Exercises

For the following problems, next to each real number, note all collections to which it belongs by writing N for natural numbers, W for whole numbers, Z for integers, Q for rational numbers, Ir for irrational numbers, and R for real numbers. Some numbers may require more than one letter.

Exercise 14 *(Solution on p. 9.)*

$\frac{1}{2}$

Exercise 15

-12

Exercise 16 *(Solution on p. 9.)*

0

Exercise 17

$-24\frac{7}{8}$

Exercise 18 *(Solution on p. 9.)*

$86.3333\ldots$

Exercise 19

$49.125125125\ldots$

Exercise 20 *(Solution on p. 9.)*

-15.07

For the following problems, draw a number line that extends from -3 to 3. Locate each real number on the number line by placing a point (closed circle) at its approximate location.

Exercise 21

$1\frac{1}{2}$

Exercise 22 *(Solution on p. 9.)*

-2

Exercise 23

$-\frac{1}{8}$

Exercise 24 *(Solution on p. 9.)*

Is 0 a positive number, negative number, neither, or both?

Exercise 25

An integer is an even integer if it can be divided by 2 without a remainder; otherwise the number is odd. Draw a number line that extends from -5 to 5 and place points at all negative even integers and at all positive odd integers.

Exercise 26 *(Solution on p. 9.)*

Draw a number line that extends from -5 to 5. Place points at all integers strictly greater than -3 but strictly less than 4.

For the following problems, draw a number line that extends from -5 to 5. Place points at all real numbers between and including each pair of numbers.

Exercise 27

-5 and -2

Exercise 28 *(Solution on p. 9.)*

-3 and 4

Exercise 29

-4 and 0

Exercise 30 *(Solution on p. 9.)*

Draw a number line that extends from -5 to 5. Is it possible to locate any numbers that are strictly greater than 3 but also strictly less than -2?

For the pairs of real numbers shown in the following problems, write the appropriate relation symbol ($<$, $>$, $=$) in place of the $*$.

Exercise 31

$-5 * -1$

Exercise 32 *(Solution on p. 10.)*

$-3 * 0$

Exercise 33

$-4 * 7$

Exercise 34 *(Solution on p. 10.)*

$6 * -1$

Exercise 35

$-\frac{1}{4} * -\frac{3}{4}$

Exercise 36 *(Solution on p. 10.)*

Is there a largest real number? If so, what is it?

Exercise 37

Is there a largest integer? If so, what is it?

Exercise 38 *(Solution on p. 10.)*

Is there a largest two-digit integer? If so, what is it?

Exercise 39

Is there a smallest integer? If so, what is it?

Exercise 40 *(Solution on p. 10.)*

Is there a smallest whole number? If so, what is it?

For the following problems, what numbers can replace x so that the following statements are true?

Exercise 41

$-1 \le x \le 5 \quad x \, \text{an integer}$

Exercise 42 *(Solution on p. 10.)*

$-7 < x < -1, \quad x \, \text{an integer}$

Exercise 43

$-3 \le x \le 2, \quad x \, \text{a natural number}$

Exercise 44 *(Solution on p. 10.)*

$-15 < x \le -1, \quad x \, \text{a natural number}$

Exercise 45

$-5 \le x < 5, \quad x \, \text{a whole number}$

Exercise 46 *(Solution on p. 10.)*

The temperature in the desert today was ninety-five degrees. Represent this temperature by a rational number.

Exercise 47

The temperature today in Colorado Springs was eight degrees below zero. Represent this temperature with a real number.

Exercise 48 *(Solution on p. 10.)*
Is every integer a rational number?

Exercise 49
Is every rational number an integer?

Exercise 50 *(Solution on p. 10.)*
Can two rational numbers be added together to yield an integer? If so, give an example.

For the following problems, on the number line, how many units (intervals) are there between?

Exercise 51
0 and 2?

Exercise 52 *(Solution on p. 10.)*
-5 and 0?

Exercise 53
0 and 6?

Exercise 54 *(Solution on p. 10.)*
-8 and 0?

Exercise 55
-3 and 4?

Exercise 56 *(Solution on p. 10.)*
m and n, $m > n$?

Exercise 57
$-a$ and $-b$, $-b > -a$?

12 Exercises for Review

Exercise 58 *(Solution on p. 10.)*
(**here**[1]) Find the value of $6 + 3(15 - 8) - 4$.

Exercise 59
(**here**[2]) Find the value of $5(8 - 6) + 3(5 + 2 \cdot 3)$.

Exercise 60 *(Solution on p. 10.)*
(**here**[3]) Are the statements $y < 4$ and $y \geq 4$ the same or different?

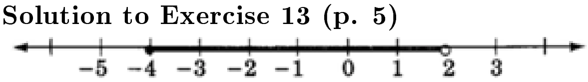
Exercise 61
(**here**[4]) Use algebraic notation to write the statement "six times a number is less than or equal to eleven."

Exercise 62 *(Solution on p. 10.)*
(**here**[5]) Is the statement $8(15 - 3 \cdot 4) - 3 \cdot 7 \geq 3$ true or false?

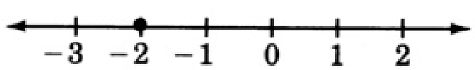
[1]"Basic Properties of Real Numbers: Symbols and Notations" <http://cnx.org/content/m18872/latest/>
[2]"Basic Properties of Real Numbers: Symbols and Notations" <http://cnx.org/content/m18872/latest/>
[3]"Basic Properties of Real Numbers: Symbols and Notations" <http://cnx.org/content/m18872/latest/>
[4]"Basic Properties of Real Numbers: Symbols and Notations" <http://cnx.org/content/m18872/latest/>
[5]"Basic Properties of Real Numbers: Symbols and Notations" <http://cnx.org/content/m18872/latest/>

Solutions to Exercises in this Module

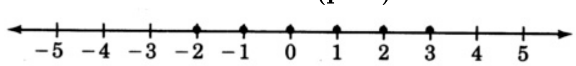
Solution to Exercise 1 (p. 4)
yes
Solution to Exercise 2 (p. 4)
yes
Solution to Exercise 3 (p. 4)
yes
Solution to Exercise 4 (p. 4)
yes
Solution to Exercise 5 (p. 4)
no
Solution to Exercise 6 (p. 4)
yes
Solution to Exercise 7 (p. 4)
yes
Solution to Exercise 8 (p. 4)
yes
Solution to Exercise 9 (p. 4)
yes
Solution to Exercise 10 (p. 5)
no, no
Solution to Exercise 11 (p. 5)
infinitely many, infinitely many
Solution to Exercise 12 (p. 5)
0, 1, 2
Solution to Exercise 13 (p. 5)



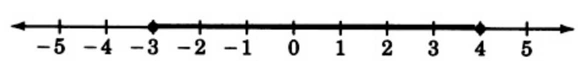
Solution to Exercise 14 (p. 6)
Q, R
Solution to Exercise 16 (p. 6)
W, Z, Q, R
Solution to Exercise 18 (p. 6)
Q, R
Solution to Exercise 20 (p. 6)
Q, R
Solution to Exercise 22 (p. 6)



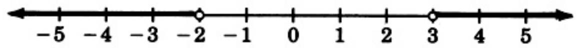
Solution to Exercise 24 (p. 6)
neither
Solution to Exercise 26 (p. 6)



Solution to Exercise 28 (p. 6)



Solution to Exercise 30 (p. 7)



; no

Solution to Exercise 32 (p. 7)

$<$

Solution to Exercise 34 (p. 7)

$>$

Solution to Exercise 36 (p. 7)

no

Solution to Exercise 38 (p. 7)

99

Solution to Exercise 40 (p. 7)

yes, 0

Solution to Exercise 42 (p. 7)

$-6, \ -5, \ -4, \ -3, \ -2$

Solution to Exercise 44 (p. 7)

There are no natural numbers between -15 and -1.

Solution to Exercise 46 (p. 7)

$\left(\frac{95}{1}\right)^{\circ}$

Solution to Exercise 48 (p. 8)

Yes, every integer is a rational number.

Solution to Exercise 50 (p. 8)

Yes. $\frac{1}{2}+\frac{1}{2}=1\text{or}1+1=2$

Solution to Exercise 52 (p. 8)

5 units

Solution to Exercise 54 (p. 8)

8 units

Solution to Exercise 56 (p. 8)

$m-n$ units

Solution to Exercise 58 (p. 8)

23

Solution to Exercise 60 (p. 8)

different

Solution to Exercise 62 (p. 8)

true